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# Effects of wavelength ratio on wave modelling

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The efficacy of perturbation approaches for short–long wave interactions is examined by considering a simple case of two interacting wave trains with different wavelengths. Frequency-domain solutions are derived up to third order in wave steepness using two different formulations: one employing conventional wave-mode functions only, and the other introducing a *modulated* wave-mode representation for the short-wavelength wave. For long-wavelength wave steepness and short-to-long wavelength ratio $\epsilon_1$ and $\epsilon_3$ respectively, the two results are shown to be identical for $\epsilon_1 \ll \epsilon_3 < 0.5$. As $\epsilon_1$ approaches $\epsilon_3$, the conventional wave-mode approach converges slowly and eventually diverges for $\epsilon_1 \gg \epsilon_3$. The loss of convergence is because the linear phase of conventional wave-mode functions is ineffective for modelling the modulated phase of the short wave. As expected, this difficulty can be removed by using a modulated wave-mode function for the short wave. On the other hand, for relatively large $\epsilon_3 \sim O(1)$, the conventional wave-mode approach converges rapidly while the slowly varying interaction between the two waves cannot be accurately predicted by the present modulated wave-mode approach. These findings have important implications to (time-domain) numerical simulations of the nonlinear evolution of ocean wave fields, and suggest that a hybrid wave model employing both conventional (for large-$\epsilon_3$ interactions) and modulated (for small-$\epsilon_3$ interactions) wave-mode functions should be particularly effective.

## 1. Introduction

Conventional frequency-domain perturbation approaches have been successfully and widely used to solve a variety of nonlinear wave dynamic problems, such as short- and long-wave interactions (Longuet-Higgins & Stewart 1960) and nonlinear wave energy transfer (Hasselmann 1962). In these perturbation approaches, the potential in the free-surface boundary conditions is expanded about the calm water level ($z = 0$). When such an approach is used for interacting waves with disparate wavelength scales, however, a convergence difficulty may be encountered as noted by Holliday (1977). He thought that the difficulty was due to the expansion about $z = 0$ but his arguments did not find universal acceptance. In a response to criticisms that high-order mode-coupling numerical schemes were incapable of describing the interaction between long and short waves. Brueckner & West (1988) argued that even though the expansion of the potential about a reference surface (say, $z = 0$) diverges when truncated at finite order, the boundary conditions at the free surface are still well behaved. They showed that the divergent terms (resulting from the expansion about $z = 0$) cancel in the free-surface boundary conditions for two waves separated in scale. Furthermore, they suggest that even conventional frequency-domain perturbations may be free of convergence difficulty.

In this paper, we revisit this controversy to investigate the effects of wavelength ratio on the convergence of conventional frequency-domain perturbations. We confirm that the convergence difficulty in the perturbation is independent of the expansion about the calm water level. However, we also find that slow convergence may result from nonlinear terms, such as the velocity product of waves with disparate wavelength scales in the free-surface dynamic condition. This difficulty can be eliminated if the modulation of short-wave phases is properly modelled.

Conventional wave-mode functions assume that the wavenumber and phase of each wave mode can be modelled by a constant and a linear function, respectively (for example $a\,e^{i(kx-\sigma t)}$ and $A\,e^{(kz+i(kx-\sigma t))}$ respectively for the elevation and velocity potential of a two-dimensional wave in deep water). When short and long waves interact with each other, the characteristics of the short waves are modulated by the presence of long waves. Specifically, a short wave riding on a long wave becomes shorter in wavelength and larger in amplitude at the crest of the long wave, and longer and smaller at the trough of the long wave. Since conventional wave-mode functions do not explicitly account for such modulations, the changes in the short-wave wavelength and amplitude along the long wave must be implicitly described, for example by second-order wave-mode functions (Longuet-Higgins & Stewart 1960). When the solution is truncated at finite order, rapid convergence is possible only if the wavelength ratio of short to long wave ($\epsilon_3$) is relatively large. When this ratio is smaller than the long-wave steepness ($\epsilon_1$), the second-order solution describing the short-wave modulation is greater than the leading-order short-wave solution, which causes convergence difficulty.

Recently, an innovative perturbation approach was developed by Phillips (1981) and Longuet-Higgins (1987) for investigating a linear short wave riding on a much longer wave. This approach was extended by Zhang & Melville (1990) to study the evolution of weakly nonlinear narrowband short gravity waves riding on a finite-amplitude periodic long wave. The key difference between this latter approach and conventional perturbation is in the modelling of the short waves. Instead of conventional wave-mode functions, the short waves are described by *modulated* wave-mode functions, which define the 'calm' water level of the short waves at the (undisturbed) surface of the long wave. More importantly, the modulation of the short-wave phases is now expressed explicitly. It is shown that, when $\epsilon_3 \ll \epsilon_1$, the modulated short-wave phase can differ significantly (of $O(\epsilon_1 \epsilon_3^{-1})$) from its average phase, that is the linear phase in the corresponding conventional wave-mode function. This large difference in phase is found to be responsible for the convergence difficulty in conventional perturbation approaches.

In this paper, the interaction between two waves with different wavelengths is studied as a simple example for general short–long wave interactions. Two different frequency-domain perturbation approaches are presented and compared: one using conventional wave-mode functions exclusively, and the other using a modulated wave-mode function for the short wave. The governing equations for the two approaches are given in §2. Solutions up to third order in wave steepness derived by the two approaches are given respectively in §3 and §4. In §5, their solutions are compared and shown to be consistent. For different values of $\epsilon_3$, however, the rates of convergence of the two approaches are quite different and reveal the main reason for the convergence difficulty in the conventional approach. The implications of this finding for mode-coupling numerical schemes are explored.

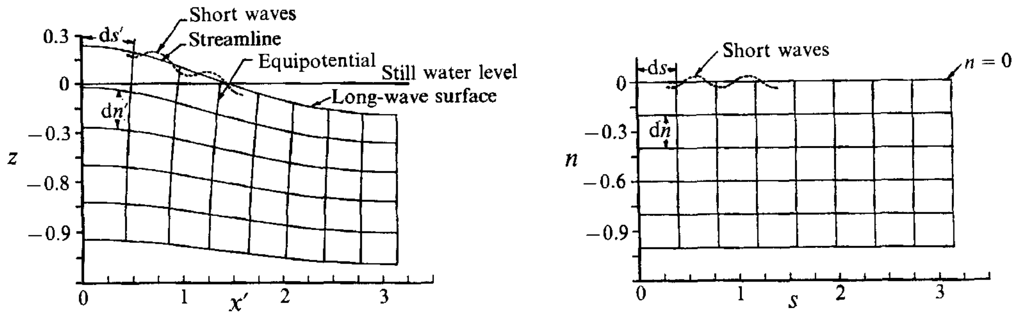


FIGURE 1. Sketch of the conformal mapping.

## 2. Formulation

We consider two-dimensional weakly nonlinear gravity wave trains with different wavelengths but propagating in the same direction in deep water. It is assumed that the flow is incompressible and irrotational and that the pressure is constant at the free surface. The rectilinear coordinates $(x, z)$ are fixed in space with the $x$-axis pointing in the direction of wave propagation and $z = 0$ is located at the calm water level. The governing equations in the $(x, z)$ coordinates are

$$\nabla^2(\Phi+\phi) = 0, \tag{2.1}$$

$$\frac{\partial \Phi}{\partial t}+\frac{\partial \phi}{\partial t}+\tfrac{1}{2}|\nabla(\Phi+\phi)|^2+g(\eta+\zeta) = 0 \quad \text{at} \quad z = \eta+\zeta, \tag{2.2}$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial \zeta}{\partial t}+\left(\frac{\partial \Phi}{\partial x}+\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}+\frac{\partial \zeta}{\partial x}\right) = \frac{\partial \Phi}{\partial z}+\frac{\partial \phi}{\partial z} \quad \text{at} \quad z = \eta+\zeta, \tag{2.3}$$

$$\nabla \Phi \rightarrow 0 \quad \text{and} \quad \nabla \phi \rightarrow 0 \quad \text{when} \quad z \downarrow -\infty, \tag{2.4}$$

where $\Phi, \phi$ and $\eta, \zeta$ are respectively the wave potentials and elevations of the long and short waves.

We introduce the orthogonal curvilinear coordinates $(s, n)$ (see Zhang & Melville 1990) which are related to the rectilinear coordinates $(x' = x+Ct, z)$ moving at the long-wave phase velocity, $|C|$, through the conformal mapping, defined by

$$s = \Phi'/C, \quad n = \Psi'/C, \tag{2.5}$$

where $\Phi'$ and $\Psi'$ are the potential and stream function of the long wave in the moving coordinates, and hence are steady. The conformal mapping projects the horizontal and vertical lines in the $(s, n)$-plane onto the streamlines and equipotentials of the long wave in the $(x', z)$-plane, and in particular, $n = 0$ coincides with the long-wave surface (see figure 1). The transformations between $(s, n)$ and $(x, z)$ are given by

$$x+Ct = s+a_1\,e^{Kn}\sin Ks+\epsilon_1\,a_1\,e^{2Kn}\sin 2Ks+O(\epsilon_1^2)\,a_1, \tag{2.6a}$$

$$z = n-\tfrac{1}{2}\epsilon_1\,a_1+a_1\,e^{Kn}\cos Ks+\epsilon_1\,a_1\,e^{2Kn}\cos 2Ks+O(\epsilon_1^2)\,a_1, \tag{2.6b}$$

$$s = x+Ct-a_1\,e^{Kz}\sin \Theta+O(\epsilon_1^2)\,a_1, \tag{2.7a}$$

$$n = z+\tfrac{1}{2}a_1\,\epsilon_1-a_1\,e^{Kz}\cos \Theta+O(\epsilon_1^2)\,a_1. \tag{2.7b}$$

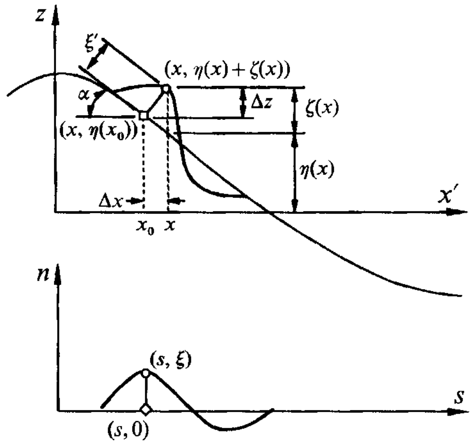


FIGURE 2. Sketch of the short-wave elevation defined in the $(x', z)$- and $(s, n)$-planes.

For $n = 0$,
$$s = x+Ct-a_1 \sin \Theta - \tfrac{1}{2}a_1 \epsilon_1 \sin 2\Theta + O(\epsilon_1^2)\, a_1. \tag{2.7c}$$

Here $a_1$, $K$, $\Omega$, and $\Theta = Kx-\Omega t$ are respectively the amplitude, wavenumber, frequency and phase of the long wave. The small parameters $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ representing the long- and short-wave steepnesses respectively and the wavelength ratio of short to long wave are defined by

$$\epsilon_1 = a_1 K, \quad \epsilon_2 = a_2 k, \quad \epsilon_3 = K/k, \tag{2.8}$$

where $a_2$ and $k$ are the average amplitude and wavenumber of the short wave.

The corresponding governing equations in $(s, n)$ coordinates are

$$\frac{\partial^2 \phi}{\partial s^2}+\frac{\partial^2 \phi}{\partial n^2} = 0, \quad -\infty < n < \xi, \tag{2.9}$$

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\left[\left(U+H\frac{\partial \phi}{\partial s}\right)^2+H^2\left(\frac{\partial \phi}{\partial n}\right)^2\right]+g\eta+g\xi' \cos\alpha = C_0 \quad \text{at} \quad n = \xi, \tag{2.10}$$

$$\frac{\partial \xi}{\partial t}+H\frac{\partial \xi}{\partial s}\left(U+H\frac{\partial \phi}{\partial s}\right) = H^2\frac{\partial \phi}{\partial n} \quad \text{at} \quad n = \xi, \tag{2.11}$$

$$U \rightarrow C, \quad \partial\phi/\partial n \rightarrow 0 \quad \text{as} \quad n \rightarrow -\infty, \tag{2.12}$$

$$\xi' = \int_0^{\xi} \frac{\mathrm{d}n}{H}, \tag{2.13}$$

where $\xi$ and $\xi'$ denote the short-wave elevations measured in the $(s, n)$ and $(x', z)$ coordinates, respectively. As shown in figure 2, $\xi'$ is measured normal to the long-wave surface and hence is different from $\zeta$. Here $\alpha$ is the local slope of the long-wave surface, $U$ the velocity induced by the long wave in the moving coordinates $(x', z)$, $C_0$ the Bernoulli constant, and $H$ the scale factor between the coordinates $(s, n)$ and $(x', z)$:

$$H(s, n) = \left[\left(\frac{\partial x'}{\partial s}\right)^2+\left(\frac{\partial z}{\partial s}\right)^2\right]^{-\frac{1}{2}}. \tag{2.14}$$

The derivations of the governing equations in the $(s, n)$-plane and (2.14) are outlined in Appendix A.

## 3. Solution by the conventional perturbation approach

Since conventional wave-mode potential functions satisfy the Laplace equation and the bottom boundary condition, only the free-surface boundary conditions need to be further imposed. Longuet-Higgins & Stewart (1960) gave a typical conventional perturbation approach and derived the solution up to second order in wave steepness. Following them, we extend the solution to third order. For brevity, detail derivations are omitted.

The leading-order solutions for the long and short waves are:

$$\Phi^{(1)} = A_1 \, \mathrm{e}^{Kz} \sin \Theta, \quad \eta^{(1)} = a_1 \cos \Theta, \tag{3.1a, b}$$

$$\phi^{(1)} = A_2 \, \mathrm{e}^{kz} \sin \theta, \quad \zeta^{(1)} = a_2 \cos \theta, \tag{3.2a, b}$$

where

$$\theta = kx - \sigma t + \beta; \quad a_1 = A_1 \Omega / g; \quad a_2 = A_2 \sigma / g, \tag{3.3a–c}$$

and $\theta$ is the short-wave phase. $A_1$ and $A_2$ are the potential amplitudes of the long and short waves, and $\sigma$ the average frequency of the short wave. Without the loss of generality, the initial phases of the long and short waves have been set to zero and $\beta$ respectively.

The second-order solution is given by

$$\Phi^{(2)} + \phi^{(2)} = -a_1 k A_2 \, \mathrm{e}^{(k-K)z} \sin(\theta - \Theta), \tag{3.4}$$

$$\eta^{(2)} + \zeta^{(2)} = \tfrac{1}{2} a_1^2 K \cos 2\Theta + \tfrac{1}{2} a_2^2 k \cos 2\theta + \epsilon_1 a_2 \cos \Theta \cos \theta - \epsilon_1 \epsilon_3^{-1} a_2 \sin \Theta \sin \theta. \tag{3.5}$$

In addition to the second harmonics of the long- and short-wave elevations, the second-order result contains wave–wave interaction terms. When $K \ll k$ (that is $\epsilon_3 \ll 1$), they describe the modulation of the short wave by the long wave. Combining them with the leading-order short-wave solution, the modulation of the short wave along the long wave can be explicitly described by

$$\tilde{\phi} = A_2 \, \mathrm{e}^{\tilde{k}(z - \eta^{(1)})} \sin \tilde{\theta}, \tag{3.6}$$

$$\tilde{\zeta} = \tilde{a}_2 \cos \tilde{\theta}, \tag{3.7}$$

where $\tilde{a}_2 = a_2(1 + \epsilon_1 \cos \Theta)$, $\tilde{\theta} = \theta + k a_1 \sin \Theta$, and $\tilde{k} = \partial \tilde{\theta} / \partial x = k(1 + \epsilon_1 \cos \Theta)$.

The modulation of the short-wave elevation was first quantified by Longuet-Higgins & Stewart (1960), and later confirmed by a direct model of the short-wave modulation at the long-wave surface (Phillips 1981). The modulation of the potential amplitude is negligible in comparison with the elevation amplitude owing to the cancellation between the modulation of the elevation amplitude and the change in the gravitational acceleration field in the presence of the long wave. Although the changes of the modulated wavenumber and elevation amplitude with respect to their average values are $O(\epsilon_1)$, the change of the modulated phase is $O(\epsilon_1 \epsilon_3^{-1})$. When this is greater than unity, it is shown that the explicit description of the modulation cannot be obtained through (3.1)–(3.5).

The magnitude ratio of the second-order solution to the leading-order short-wave solution is $O(a_1 k)$, which is the same as that of the phase modulation. This coincidence is by no means accidental, as we will show. If the solution is truncated at second order, for convergence, we require that $\epsilon_1 \ll \epsilon_3$.

The average frequencies of the short and long waves up to third order are given by

$$\sigma^2 = gk[1+a_2^2k^2+2a_1^2Kk\Omega/\sigma], \tag{3.8}$$

$$\Omega^2 = gK[1+a_1^2K^2+2a_2^2Kk\Omega/\sigma]. \tag{3.9}$$

The second terms in the brackets of both equations are due to the nonlinearity of the wave itself, while the third terms result from the presence of the other wave. Equations (3.8) and (3.9) are in agreement with Longuet-Higgins & Phillips (1962) and describe the average absolute frequencies instead of intrinsic frequencies.

The third-order potential is given by

$$\Phi^{(3)}+\phi^{(3)} = \tfrac{1}{2}a_2^2K^2A_1\frac{3-2(\Omega/\sigma)}{(1-\Omega/\sigma)^2}\mathrm{e}^{(2k-K)z}\sin(2\theta-\Theta)+\tfrac{1}{2}a_1^2k^2A_2T_1\mathrm{e}^{|k-2K|z}\sin(\theta-2\Theta), \tag{3.10a}$$

where

$$T_1 = \begin{cases} 1-2\epsilon_3 & \text{if } k>2K, \\ (\Omega/\sigma)(2\epsilon_3-1)(2-3\Omega/\sigma)/(1-\Omega/\sigma)^2 & \text{if } k\leqslant 2K. \end{cases} \tag{3.10b}$$

When $\epsilon_3 \ll 1$, the last term in (3.10$a$) is dominant. For convergence, we require $\epsilon_1 \ll 2\epsilon_3$. In our later comparison with the solution from the modulated wave-mode approach, we will show that the ratio of two consecutive orders of the solution is approximately $(\epsilon_1\epsilon_3^{-1})/(m-1)$, where $m$ is the perturbation order.

The third-order elevation is given by

$$\begin{aligned}\eta^{(3)}+\zeta^{(3)} = &-\tfrac{3}{8}a_1^3K^2\cos\Theta+\tfrac{3}{8}a_1^3K^2\cos 3\Theta-\tfrac{3}{8}a_2^3k^2\cos\theta+\tfrac{3}{8}a_2^3k^2\cos 3\theta \\ &+a_1a_2^2Kk(\tfrac{1}{4}\epsilon_3-\Omega/\sigma)\cos\Theta+a_1^2a_2k^2[\epsilon_3(\tfrac{1}{2}-\Omega/\sigma)-\tfrac{1}{4}]\cos\theta \\ &+a_1a_2^2Kk(\tfrac{1}{4}\epsilon_3+\tfrac{3}{2}+T_2)\cos\Theta\cos 2\theta-a_1a_2^2k^2(1-\tfrac{1}{2}\epsilon_3-\epsilon_3T_2)\sin\Theta\sin 2\theta \\ &+a_1^2a_2k^2T_3\cos 2\Theta\cos\theta+a_1^2a_2k^2T_4\sin 2\Theta\sin\theta, \end{aligned} \tag{3.11a}$$

where

$$T_2 = 2\frac{\Omega}{\sigma}+\frac{5}{2}\frac{(\Omega/\sigma)^2}{(1-\Omega/\sigma)}+\frac{1}{2}\frac{(\Omega/\sigma)^3}{(1-\Omega/\sigma)^2}, \qquad T_3 = \begin{cases} \tfrac{1}{4}+\epsilon_3^2 & \text{if } k>2K, \\ -\tfrac{1}{4}+\epsilon_3+\epsilon_3^2+T_5 & \text{if } k\leqslant 2K, \end{cases} \tag{3.11b, c}$$

$$T_4 = \begin{cases} -\epsilon_3 & \text{if } k>2K, \\ -\tfrac{1}{2}+T_5 & \text{if } k\leqslant 2K, \end{cases} \qquad T_5 = (\tfrac{1}{2}-\epsilon_3)\left[3\frac{(\Omega/\sigma)^2}{(1-\Omega/\sigma)}-\frac{(\Omega/\sigma)^3}{(1-\Omega/\sigma)^2}\right]. \tag{3.11d, e}$$

The first four terms in (3.11$a$) show the first and third harmonics of the short and long waves, respectively. The fifth term gives the changes of the long-wave elevation due to the presence of the short wave and the remaining terms describe the modulation of the short-wave elevation by the long wave. When $\epsilon_3 \ll 1$, the third-order elevation is dominated by

$$\eta^{(3)}+\zeta^{(3)} \approx -\tfrac{1}{2}a_1^2a_2k^2\sin^2\Theta\cos\theta. \tag{3.12}$$

Thus, the requirement for the convergence of the wave elevation is the same as that for (3.10$a$), i.e. $\epsilon_1 \ll 2\epsilon_3$.

One may disagree with the criteria of convergence used herein and argue that while the second-order solution may be larger than the leading-order short wave, it is always smaller than the leading-order long wave, and hence the perturbation may be considered convergent when truncated at finite order. The reason why the

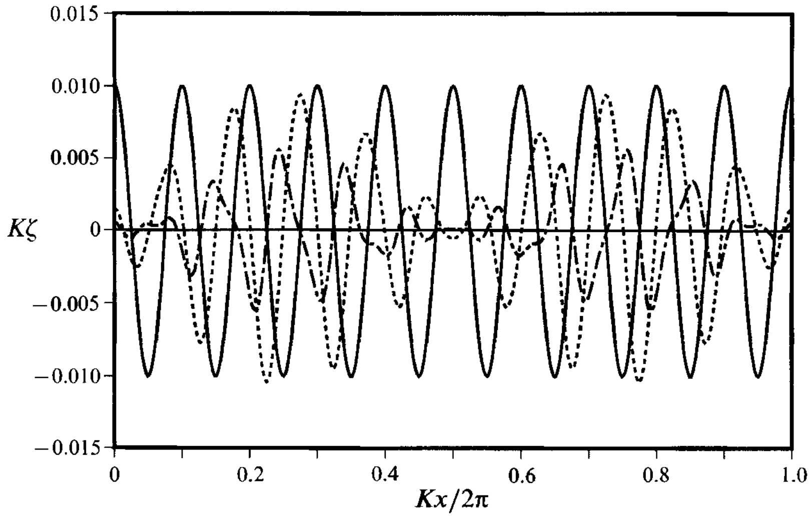


FIGURE 3. The second-order (----) and third-order (———) solutions for the wave elevation given by (3.5) and (3.11*a*), respectively, are compared with the leading-order short-wave elevation (——), for $\epsilon_1 = 0.1$, $\epsilon_2 = 0.1$ and $\epsilon_3 = 0.1$.

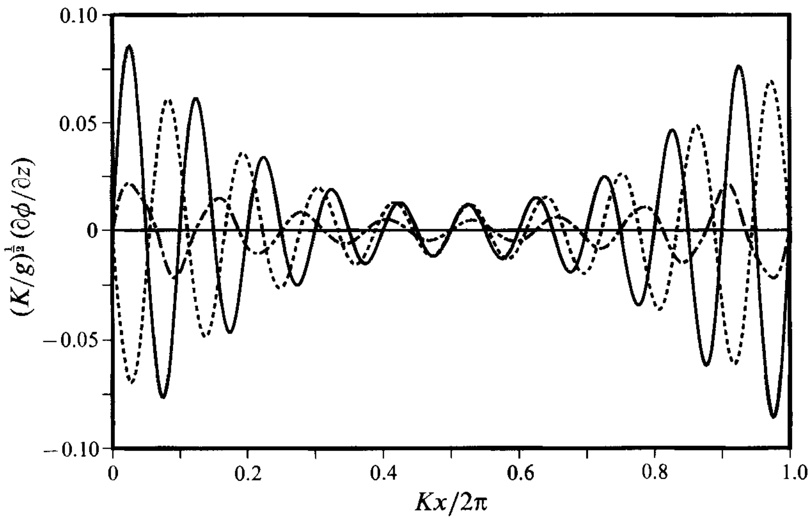


FIGURE 4. The second-order (----) and third-order (———) solutions for the vertical velocity based on (3.5) and (3.10*a*), respectively, are compared with the leading-order short-wave vertical velocity (——), for the same $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ as figure 3.

comparisons are made with the leading-order short wave instead of the long wave is elaborated below.

When $K \ll k$, the wavenumbers and frequencies of the dominant second- and third-order (interaction) terms are close to those of the short wave, and the high-order solutions behave like the short-wave solution. This is demonstrated clearly in figures 3 and 4 for $\epsilon_1 = 0.1$, $\epsilon_2 = 0.1$ and $\epsilon_3 = 0.1$, where the leading-order short-wave elevation and vertical velocity (at the long-wave surface) are compared with those of the second- and third-order solutions respectively. Thus, the high-order solutions modify essentially the short wave rather than the long wave, as depicted in

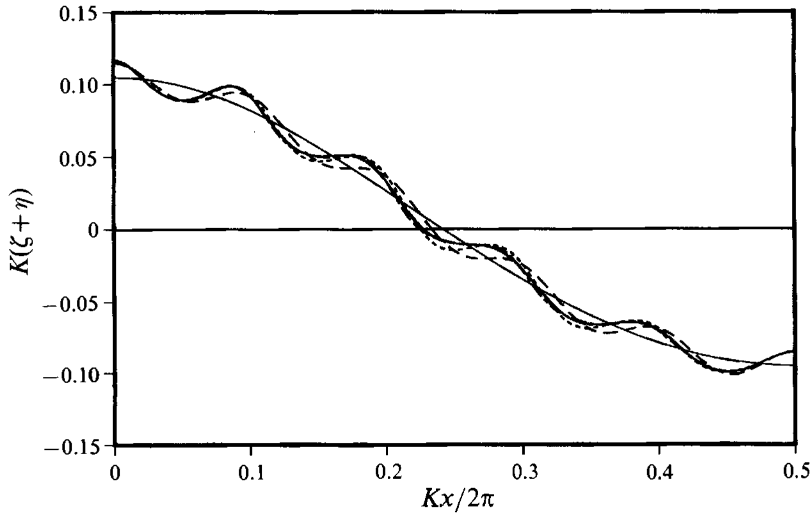


FIGURE 5. The resultant wave elevations of a long wave (up to the third-order) and a short wave up to the leading order (– – –), the second order (- - - -) and the third order (——— ) are compared with that obtained by the modulated wave-mode approach (——), for the same $\epsilon_1$, $\epsilon_2$ as figure 3. For reference, the undisturbed long-wave surface (——) up to the third order is also plotted. Taking advantage of the symmetry, we only show the resultant wave elevation along a half long wavelength.

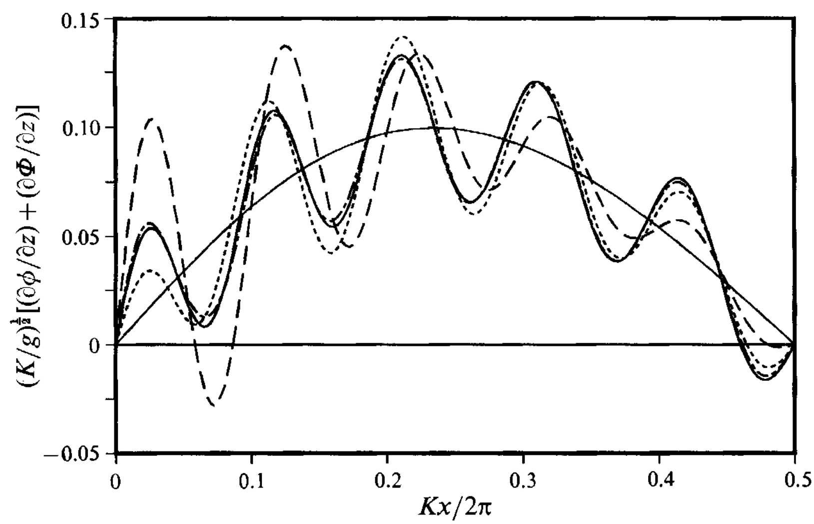


FIGURE 6. The resultant vertical velocities of a long wave and a short wave up to the leading order (– – –), the second order (- - - -) and the third order (——— ) are compared with that obtained by the modulated wave-mode approach (——), for the same $\epsilon_1$, $\epsilon_2$ as figure 3. For reference, the undisturbed long-wave vertical velocity (——) up to the third order is also plotted. Taking advantage of the antisymmetry, we only show the resultant vertical velocity along a half long wavelength.

figures 5 and 6. The corresponding results using the modulated wave-mode approach are also plotted in these figures as asymptotes of the conventional perturbation solution when truncated at very high orders $m$ (that is when $(\epsilon_1/\epsilon_3)^{(m-1)}/(m-1)! \ll 1$). It is seen that the curves that include up to third-order terms approach the

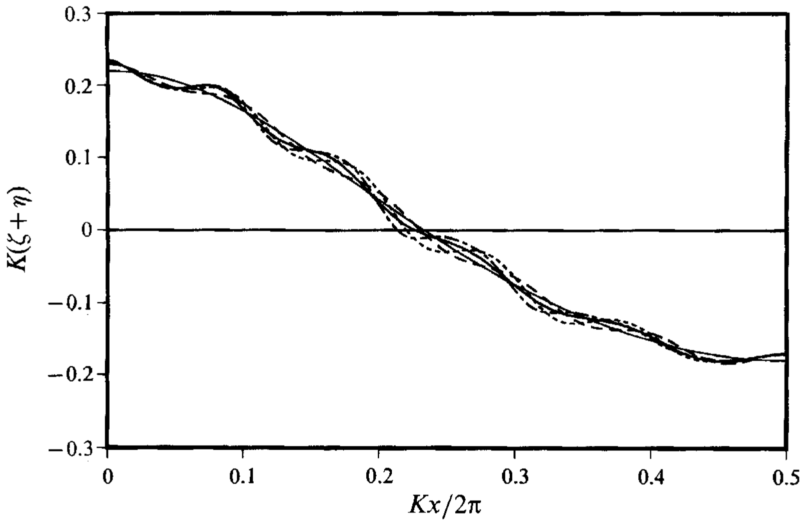


FIGURE 7. Same as figure 5, except $\epsilon_1 = 0.2$.

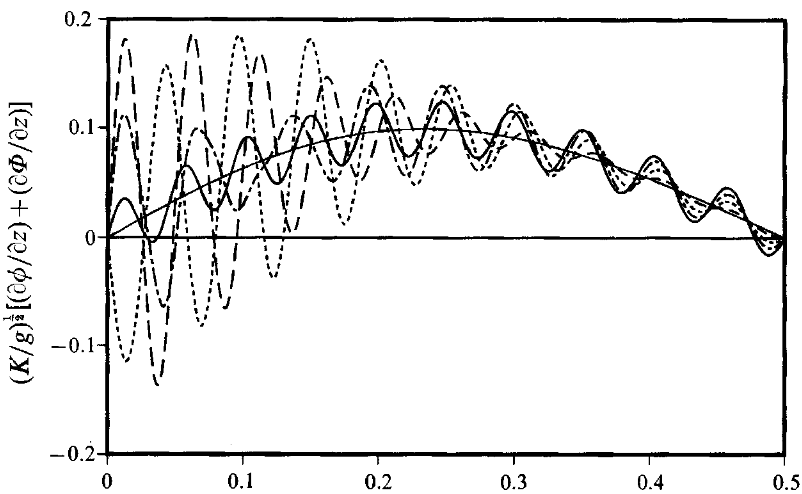


FIGURE 8. Same as figure 6, except $\epsilon_3 = 0.05$.

asymptotes, while those that include up to the second-order solutions are in fact farther away from the asymptotes than the leading-order solutions. This indicates the divergence of the solution when truncated at second order. When $\epsilon_1/\epsilon_3$ increases, the solution may fail to converge even if truncated at relatively higher order. This trend is observed in figures 7 and 8, where the resultant wave elevation for larger long-wave steepness ($\epsilon_1 = 0.2$) and the resultant vertical velocity for smaller wavelength ratio ($\epsilon_3 = 0.05$) are displayed respectively. The differences between the curves that include up to the third-order solution and asymptotes are significantly greater than those shown in figures 5 and 6, which indicates that the solution does not converge if truncated at the third order.

It is also important to show that the convergence difficulty is not caused by the expansion at $z = 0$. To demonstrate this, we revisit the two-wave interaction problem using the two-dimensional Zakharov equation, which applies at the free surface:

$$\frac{\partial(\Phi+\phi)^{(S)}}{\partial t}+\frac{1}{2}\left[\frac{\partial(\Phi+\phi)^{(S)}}{\partial x}\right]^2+g(\eta+\zeta)=\frac{1}{2}\left\{1+\left[\frac{\partial(\eta+\zeta)}{\partial x}\right]^2\right\}(W^{(S)})^2, \qquad (3.13a)$$

$$\frac{\partial(\eta+\zeta)}{\partial t}+\frac{\partial(\Phi+\phi)^{(S)}}{\partial x}\frac{\partial(\eta+\zeta)}{\partial x}=\left\{1+\left[\frac{\partial(\eta+\zeta)}{\partial x}\right]^2\right\}W^{(S)}, \qquad (3.13b)$$

where the superscript S denotes the value defined at the free surface, and $W$ the vertical velocity. Following a similar perturbation approach as above except that no expansion about $z=0$ is performed, we obtain the solution for the potential and wave elevation ($W$ can be determined from the surface potential and elevation following Watson & West 1975, and West 1981). Up to the second order, it is given by

$$(\Phi+\phi)^{(S)(1)}=\frac{\Omega}{K}a_1\sin\Theta+\frac{\sigma}{k}a_2\sin\theta, \qquad (3.14a)$$

$$(\eta+\zeta)^{(1)}=a_1\cos\Theta+a_2\cos\theta, \qquad (3.14b)$$

$$(\Phi+\phi)^{(S)(2)}=\tfrac{1}{2}\Omega a_1^2\sin 2\Theta+\tfrac{1}{2}\sigma a_2^2\sin 2\theta+(\Omega+\sigma)\,a_1\,a_2\sin\Theta\cos\theta, \qquad (3.15a)$$

$$(\eta+\zeta)^{(2)}=\tfrac{1}{2}a_1^2K\cos 2\Theta+\tfrac{1}{2}a_2^2\,k\cos 2\theta+a_1\,a_2\,K\cos\theta\cos\Theta-a_1\,a_2\,k\sin\theta\sin\Theta. \qquad (3.15b)$$

Although the leading-order solution for the potential is deliberately chosen not to involve the large exponential factor $e^{k(\eta+\zeta)}$, the solution up to the second order is still identical to that given in (3.1)–(3.5). Hence, the convergence difficulty is not related to the expansion of the free-surface boundary condition at $z=0$.

## 4. Solution by the modulated wave-mode approach

The influence of the short wave on the long wave is expected to be of third order at most, and is much smaller than the influence of the long wave on the short wave when $\epsilon_3 \ll 1$. Therefore, changes in the long wave due to the short wave and the modulation of the short wave by the long wave can be calculated separately at least up to third order in wave steepness. Expanding (2.10) and (2.11) about $n=0$, and then subtracting the steady solution for the long wave, we derive the expanded boundary conditions (4.1) and (4.2), which together with (2.9), (2.12) constitute the governing equations for the modulated short wave:

$$\phi_t+H_0^2C\phi_s+g_1\,\xi+(\phi_t+H_0^2C\phi_s)_n\,\xi+\tfrac{1}{2}H_0^2(\phi_s^2+\phi_n^2)+\tfrac{1}{2}(\phi_t+H_0^2C\phi_s)_{nn}\,\xi^2$$
$$+\tfrac{1}{2}H_0^2(\phi_s^2+\phi_n^2)_n\,\xi=O(\epsilon_{1,2}^3)\,A_2'\,\sigma \quad \text{at} \quad n=0, \qquad (4.1)$$

$$\xi_t+H_0^2C\xi_s-H_0^2\phi_n+H_0^2(\phi_s\,\xi)_s+H_0\frac{\partial H_0}{\partial n}C(\xi^2)_s-2H_0\frac{\partial H_0}{\partial n}\xi\phi_n+\tfrac{1}{2}H_0^2(\phi_{sn}\,\xi^2)_s$$
$$=O(\epsilon_{1,2}^3)\,a_2'\,\sigma \quad \text{at} \quad n=0, \qquad (4.2)$$

where
$$H_0=H(s,0),\quad \frac{\partial H_0}{\partial n}=\left.\frac{\partial H}{\partial n}\right|_{n=0},\quad g_1=\frac{g\cos\alpha}{H_0}+H_0\frac{\partial H_0}{\partial n}C^2. \qquad (4.3a\text{–}c)$$

In the above, $g_1$ denotes the effective gravitational acceleration divided by $H_0$, and $A_2'$ and $a_2'$ the amplitudes of the short-wave potential and elevation respectively. $A_2'$ and $a_2'$ are slightly different from their counterparts $A_2$ and $a_2$, as will be shown in §5.

The subscripts $t$, $s$ and $n$ denote derivatives with respect to them. The notation $O(\epsilon^3_{1,2})$ represent the cubic products of $\epsilon_1$, $\epsilon_2$ or their combinations. In expanding the free-surface boundary conditions about $n = 0$, it is assumed that the long-wave velocity field can be analytically extended into the region between the short-wave crests and the long-wave surface.

The solution up to the third order was derived by Zhang & Melville (1990) and Zhang (1991) using perturbation and variational principle approaches, respectively. In these derivations, the instability of the short wave was considered and $\epsilon_3$ was limited to be $\ll 1$. In the present study, we relax the requirement that $\epsilon_3 \ll 1$ and focus on the steady solution of the modulated short wave. Using standard perturbation procedures, the solutions for the short and long waves are derived. Again, the details are omitted for brevity.

The first-harmonic potential and elevation of the short wave up to third order are given by

$$\phi^{(1)} = A'_2 \mathrm{e}^{k'n} \sin\theta' + \mathscr{L} + O(\epsilon^3_{1,2}) A'_2, \tag{4.4}$$

$$\xi^{(1)} = a'_2 \cos\theta' - \tfrac{3}{8} a'^3_2 k'^2 \cos\theta' + O(\epsilon^3_{1,2})\, a'_2, \tag{4.5}$$

where $\theta'$ is the phase of the short wave in the $(s, n)$ coordinates,

$$\partial\theta'/\partial s = k', \quad \partial\theta'/\partial t = -\omega, \tag{4.6}$$

$$\omega = \sigma_{\mathrm{I}} + H_0^2 C k', \tag{4.7}$$

$$\sigma^2_{\mathrm{I}} = g_1 H_0^2 k' (1 + a'^2_2 k'^2), \tag{4.8}$$

and $k'$, $\omega$, $\sigma_{\mathrm{I}}$ are respectively the wavenumber, absolute frequency (in the moving coordinates) and intrinsic frequency of the short wave. Equation (4.8) describes the nonlinear dispersion relation. The amplitudes of the potential and elevation are related by

$$a'_2 = A'_2 \sigma_{\mathrm{I}} / g_1. \tag{4.9}$$

The solution given in (4.4)–(4.9) is similar in form to that of Stokes waves except for an extra term $\mathscr{L}$ in (4.4). Since the short-wave wavenumber $k'$ varies slowly with $s$, the leading-order potential alone satisfies the Laplace equation only to leading order, and the higher-order correction term $\mathscr{L}$ must be included in order to satisfy the Laplace equation exactly. The derivation of $\mathscr{L}$ is given in Appendix B.

The second-harmonic potential and elevation of the short wave are given by (see Appendix C):

$$\phi^{(2)} = \epsilon_1 a'_2 k' A'_2 \frac{\Omega(\frac{3}{2} - \Omega/\sigma_{\mathrm{I}})}{\sigma_{\mathrm{I}}(1 - \Omega/\sigma_{\mathrm{I}})^2} \mathrm{e}^{(2k'-K)n} \sin(2\theta' - Ks), \tag{4.10}$$

$$\xi^{(2)} = \tfrac{1}{2} a'^2_2 k' \cos 2\theta' + \epsilon_1 a'^2_2 k' \frac{\Omega\,(2 - \frac{3}{2}\Omega/\sigma_{\mathrm{I}})}{\sigma_{\mathrm{I}}\,(1 - \Omega/\sigma_{\mathrm{I}})^2} \cos(2\theta' - Ks). \tag{4.11}$$

With the exception of the first term on the right-hand side of (4.11), the remaining solution represents the modulation of the short wave by the long wave.

The third harmonic of the short wave is given by

$$\phi^{(3)} = O(\epsilon^3_{1,2}) A'_2, \tag{4.12}$$

$$\xi^{(3)} = \tfrac{3}{8} a'^3_2 k'^2 \cos 3\theta' + O(\epsilon^3_{1,2})\, a'_2, \tag{4.13}$$

and terms resulting from wave interaction are of higher order.

Since the solution of an undisturbed long wave is well known, we need only to show the modification of the long wave due to the presence of the short wave. The forcing term having the long-wave phase in the free-surface dynamic boundary conditions (4.1) is $(\partial H_0^2/\partial n)(\partial\phi/\partial s)C\xi$, which decreases the long-wave elevation by (see Zhang & Melville 1990, Appendix A),

$$\xi^{(0)} = \frac{1}{2}\frac{|C|}{\sigma_{\mathrm{I}}}\frac{\partial H_0^2}{\partial n}a_2'^2 k'. \tag{4.14}$$

The decrease in the long-wave elevation induces the forcing term in the kinematic boundary condition (4.2), that is $H_0^2 C(\partial\xi^{(0)}/\partial s)$. Combining the two surface boundary conditions, the effect of the two forcing terms on the long-wave frequency is given by

$$\frac{\partial^2\Phi}{\partial t^2}+g\frac{\partial\Phi}{\partial z} = \frac{\partial}{\partial t}\left[\frac{1}{2}\frac{|C|}{\sigma_{\mathrm{I}}}\frac{\partial H_0^2}{\partial n}ga_2'^2 k'\right]-gH_0^2 C\frac{\partial\xi^{(0)}}{\partial s}+\mathscr{N}, \tag{4.15}$$

where $\mathscr{N}$ represents the nonlinear contribution from the long wave itself. Using (5.2) and (2.7*a–c*), we obtain the nonlinear dispersion relation for the long wave in the presence of the short wave:

$$\Omega^2 = gK(1+a_1^2K^2+2(\Omega/\sigma)\,a_2^2 kK). \tag{4.16}$$

The primes on $a_2$ and $k$ and the subscript I of $\sigma$ are dropped here because the differences are of higher order. The increase in the long-wave frequency is identical to that derived in §3. Correspondingly, the coordinates $(x', z)$ move at the increased long-wave phase velocity. Nevertheless, this increase in the long-wave phase velocity does not affect the short-wave solution at least up to the third order.

## 5. Conversion and comparison

### 5.1. *Conversion*

For comparison, it is useful to convert the solution derived using the modulated wave-mode approach in §4 in terms of conventional wave-mode functions and in the rectilinear coordinates $(x, z)$. The equations for converting the short-wave characteristics are given first. Substituting (2.6*a*) and (2.6*b*) into (2.14), the scale factor is calculated:

$$H = (1+2\epsilon_1\,e^{Kn}\cos Ks+\epsilon_1^2\,e^{2Kn}+4\epsilon_1^2\,e^{2Kn}\cos 2Ks)^{-\frac{1}{2}}+O(\epsilon_1^3). \tag{5.1}$$

At $n = 0$,

$$H_0 = (1+2\epsilon_1\cos Ks+\epsilon_1^2+4\epsilon_1^2\cos 2Ks)^{-\frac{1}{2}}. \tag{5.2}$$

At the interaction of the long-wave surface and the calm water level $H_0 = 1$.

According to (4.3*c*), the relative change of $g_1$ along the long wave is of $O(\epsilon_1^3)$. Furthermore, using the dispersion relation (4.8) and applying phase conservation to (4.7), the relative change of the intrinsic frequency $\sigma_{\mathrm{I}}$ was found to be of $O(\epsilon_{1,2}^3)$ (Zhang & Melville 1992). Therefore, both can be approximated by constants:

$$g_1 = (1+\epsilon_1^2+O(\epsilon_1^3))\,g. \tag{5.3}$$

$$\sigma_{\mathrm{I}} = [gk_0'(1+\epsilon_1^2+\epsilon_2^2)]^{\frac{1}{2}}+O(\epsilon_{1,2}^3)\,\sigma_{\mathrm{I}}, \tag{5.4}$$

where $k_0'$ is the short wavenumber at the intersection of the long-wave surface and the calm water level. Differentiating (4.8) with respect to $s$ and noticing $(1/g_1)\,\partial g_1/\partial s = O(\epsilon_1^3)$ and $(2/\sigma_{\mathrm{I}})\,\partial\sigma_{\mathrm{I}}/\partial s = O(\epsilon_{1,2}^3)$ based on (5.3) and (5.4), we obtain a simple relation between $k'$ and $H_0^2$,

$$\frac{1}{k'}\frac{\mathrm{d}k'}{\mathrm{d}s} = -\frac{1}{H_0^2}\frac{\mathrm{d}H_0^2}{\mathrm{d}s}+O(\epsilon_{1,2}^3), \tag{5.5}$$

and solve if for $k'$:

$$k' = k'_0(1+2\epsilon_1 \cos Ks + \epsilon_1^2 + 4\epsilon_1^2 \cos 2Ks) + O(\epsilon_{1,2}^3) k'_0. \tag{5.6}$$

Although $k'$ is different from its counterparts in the $(x', z)$-plane, the average of $k'$ (with respect to $s$) is equal to the average wavenumber in the $(x, z)$-plane, since the long wavelength and the number of short-wave crests or troughs within one long wavelength remain unchanged during the conformal mapping. Hence,

$$k'_0(1+\epsilon_1^2) = k, \tag{5.7}$$

which indicates that the short wavelength at the interaction of the long-wave surface and the calm water level (where $H_0 = 1$) is slightly longer than the average wavelength. Substituting (5.7) into (5.4), the intrinsic frequency up to the second order is found to be

$$\sigma_{\mathrm{I}}^2 = gk(1+a_2^2 k^2). \tag{5.8}$$

Comparing (5.8) with (3.8) gives the difference between $\sigma_{\mathrm{I}}$ and the average frequency $\sigma$, which is due to the convection by the long-wave particle velocity. The convectional effect on the short-wave frequency can also be recovered through the modulated wave-mode approach, as shown in Appendix D.

The short-wave phase is calculated as the integral of $k'$ in $s$ and $-\omega$ in time:

$$\theta' = k'_0[(1+\epsilon_1^2)s + 2a_1 \sin Ks + 2\epsilon_1 a_1 \sin 2Ks] - \omega t + \beta, \tag{5.9}$$

where $\beta$ is the initial phase. With the help of (2.7 *a*), (4.7), (3.8) and (5.7), we note that

$$k'_0(1+\epsilon_1^2)s - \omega t + \beta = \theta - ka_1 \mathrm{e}^{Kz} \sin \Theta + O(\epsilon_1^2) ka_1. \tag{5.10}$$

Thus $\theta'$ is related to $\theta$ as given below which shows that the fluctuation of the modulated phase with respect to the average phase is $O(a_1 k)$:

$$\theta' = \theta + a_1 k(2-\mathrm{e}^{Kz}) \sin \Theta + \epsilon_1 a_1 k(2-\mathrm{e}^{Kz}) \sin 2\Theta + O(\epsilon_1^2) a_1 k. \tag{5.11}$$

There are two ways to project the wave elevation from the $(s, n)$- to the $(x, z)$-plane. The first is to map the elevation $\xi(s,t)$ onto $\eta(x,t)+\zeta(x,t)$ directly, using the transform function (2.7 *a–c*). This is straightforward but involves a cumbersome computation. The second method calculates the increment between the resultant wave elevation and the long-wave elevation in the $(x, z)$-plane based on $\xi(s,t)$. The resultant wave elevation is then determined by adding the increment to the undisturbed long-wave elevation. The two methods yield identical results. For clarity, only the conversion using the latter method is given here.

As depicted in figure 2, the points $(s, \xi)$ and $(s, 0)$ project onto $(x, \eta(x,t)+\zeta(x,t))$ and $(x_0, \eta(x_0, t))$ respectively in the $(x, z)$-plane. The increment between the two points in the $(x, z)$-plane is calculated using the expansion of (2.7 *a*) and (2.7 *b*) about $n = 0$:

$$\Delta x = \xi(\epsilon_1 \sin Ks + 2\epsilon_1^2 \sin 2Ks + \tfrac{1}{2}\epsilon_1 K\xi \sin Ks) + O(\epsilon_{1,2}^3)\,\xi, \tag{5.12a}$$

$$\Delta z = \xi(1+\epsilon_1 \cos Ks + 2\epsilon_1^2 \cos 2Ks + \tfrac{1}{2}\epsilon_1 K\xi \cos Ks) + O(\epsilon_{1,2}^3)\,\xi, \tag{5.12b}$$

where $\xi$ is the sum of $\xi^{(1)}$, $\xi^{(2)}$, $\xi^{(3)}$ and $\xi^{(0)}$, which are given by (4.5), (4.11), (4.13) and (4.14), respectively. Since $\xi$ is derived for $n = 0$, the coordinate $z$ in the phase of $\xi$ is substituted by the long-wave elevation:

$$\theta'|_{n=0} = \theta + a_1 k \sin \Theta + \tfrac{1}{2} a_1 k \epsilon_1 \sin 2\Theta + O(\epsilon_1^2)\, a_1 k. \tag{5.13}$$

The conversion of the modulated wave-mode functions to interaction wave-mode functions is accomplished by expanding the sinusoidal functions of the phase difference in Taylor series:

$$\cos\theta' = \cos\theta(1-\tfrac{1}{2}a_1^2 k^2 \sin^2\Theta + \ldots) - \sin\theta(a_1 k \sin\Theta - \ldots). \tag{5.14}$$

When $\epsilon_1 > \epsilon_3$, the phase difference can be greater than unity, and hence the truncation of the expansion series at finite order $m$ may not converge unless $(a_1 k)^{(m-1)}/(m-1)! \ll 1$. In fact, the comparison of (5.14) with (3.5) and (3.12) shows that the dominant divergent terms in the conventional approach are the same as those from the phase expansion of the leading-order short-wave elevation. The coincidence reveals the source of the divergent terms in the solution of the conventional approach. Substituting (5.2) and (5.14) into (4.5), (4.11), (4.13) and (4.14), $\xi$ can be obtained. Further substituting (2.7$c$) into (5.12$a$), we may express $\Delta x$ and $\Delta z$ in terms of $(x, z)$. As shown in figure 2, the resultant wave elevation is equal to the sum of the undisturbed long-wave elevation $(\eta(x_0, t))$ and $\Delta z$. Since the horizontal coordinate of the resultant elevation is moved by $\Delta x$ due to the tilt of the long-wave surface, we accordingly translate the coordinate in the solution. Finally, we separate the resultant wave elevation into the long- and short-wave elevations by recognizing their phases:

$$\eta(x,t) = a_1(1-\tfrac{3}{8}a_1^2K^2-(\Omega/\sigma)\,a_2'^2\,kK+\tfrac{1}{4}a_2'^2K^2)\cos\Theta+\tfrac{1}{2}a_1^2K\cos 2\Theta+\tfrac{3}{8}a_1^3K^2\cos 3\Theta, \tag{5.15}$$

$$\begin{aligned}\zeta(x,t) = {}& a_2'[(1-\tfrac{1}{4}a_1^2k^2-\tfrac{3}{8}a_2'^2k^2+\epsilon_1^2)+\epsilon_1\cos\Theta+(\tfrac{1}{4}a_1^2k^2+\epsilon_1^2)\cos 2\Theta]\cos\theta\\ &+a_2'^2k\left[\tfrac{1}{2}+\epsilon_1\left(\tfrac{3}{2}+\tfrac{1}{4}\epsilon_3+\frac{\Omega}{\sigma_{\mathrm{I}}}\frac{(2-\frac{3}{2}\Omega/\sigma_{\mathrm{I}})}{(1-\Omega/\sigma_{\mathrm{I}})^2}\right)\cos\Theta\right]\cos 2\theta\\ &-a_2'a_1k(\sin\Theta+\epsilon_1\sin 2\Theta)\sin\theta\\ &+a_1k^2a_2'^2\left[-1+\epsilon_3\left(\frac{1}{2}+\frac{\Omega}{\sigma_{\mathrm{I}}}\frac{(2-\frac{3}{2}\Omega/\sigma_{\mathrm{I}})}{(1-\Omega/\sigma_{\mathrm{I}})^2}\right)\right]\sin\Theta\sin 2\theta+O(\epsilon_{1,2}^3)\,a_2'.\end{aligned} \tag{5.16}$$

During this conversion, we find that all possibly divergent terms involve the product $a_1 k$, and originate from the expansion of the difference between a modulated and an average phase. The same situation is also observed in the conversion of the potential.

The conversion of the leading-order short-wave potential is similar to that of the wave elevation, and $\epsilon_1 \ll \epsilon_3$ is also required for the truncation to be valid. Details of the $\phi^{(1)}$ conversion are in Appendix E. The final results are

$$\begin{aligned}\phi^{(1)} = {}& A_2'[(1-\tfrac{1}{2}\epsilon_1 a_1 k)\,\mathrm{e}^{kz}\sin\theta-a_1k\,\mathrm{e}^{(k-K)z}\sin(\theta-\Theta)\\ &+(\tfrac{1}{2}a_1^2k^2-\epsilon_1a_1k)\,\mathrm{e}^{(k-2K)}\sin(\theta-2\Theta)]+O(a_1^3k^3)\,A_2'.\end{aligned} \tag{5.17}$$

Since the second-harmonic potential is of $O(\epsilon_{1,2}^2)\,A_2'$, its conversion is accomplished simply by substituting the variables and wave characteristics in the $(x, z)$-plane for their counterparts in (4.10); the differences due to the substitution are at most $O(\epsilon_{1,2}^3)\,A_2$:

$$\phi^{(2)} = \epsilon_1 a_2 kA_2\frac{\Omega}{\sigma}\frac{(\frac{3}{2}-\Omega/\sigma)}{(1-\Omega/\sigma)^2}\mathrm{e}^{(2k-K)z}\sin(2\theta-\Theta). \tag{5.18}$$

The long-wave potential is the same as a single Stokes wave except for the increased frequency due to the presence of the short wave:

$$\Phi = A_1\,\mathrm{e}^{Kz}\sin\Theta, \tag{5.19}$$

where $a_1 = A_1\Omega/g$.

### 5.2. *Comparison*

For the same long- and short-wave conditions, the solutions derived by the two different approaches should be identical provided they both converge. To have the same wave condition in the two approaches, we may either let the potential be the same and then compare the wave elevation based on the potential, or *vice versa*.

Comparing (3.1 *a*) with (5.19) and (3.9) with (4.16), we see that the solutions for the long-wave potential by the two approaches are indeed identical, which justifies the use of identical notions for the long wave in both approaches. Comparisons of (3.2 *a*), (3.4) and (3.10*a*) with (5.17) and (5.18), however, show two discrepancies between the solutions for the short-wave potential. The modulated wave-mode approach fails to recover part of the third-order potential for $k < 2K$, which is present in the conventional wave-mode solution (3.10*a*). The missing solution is due to the assumption that $k \gg K$ in the modulated wave-mode approach. The inclusion of the long-wave second harmonic in the short-wave modulation implies that $k > 2K$ which excludes the case $k < 2K$ and the related solution. We anticipate that the condition $k > mK$, is required in the modulated wave-mode approach when effects of the $m$th long-wave harmonic are considered in the short-wave modulation, or more generally, in order for it to be accurate up to $O(\epsilon_1^m)$. This points out the limitation of the modulated wave-mode approach for waves with close wavelength scales.

The two solutions are otherwise identical with the exception of an extra third-order term, $-\frac{1}{2}A_2'\epsilon_1 a_1 k\,\mathrm{e}^{kz}\sin\theta$, in (5.17). Thus, for the same short-wave potential amplitude, we let

$$A_2'(1-\tfrac{1}{2}\epsilon_1 a_1 k) = A_2. \tag{5.20}$$

Comparison of (5.15) with (3.1 *b*), (3.5) and (3.11 *a*) shows that the solutions for the long-wave elevation by the two approaches are identical. The last two terms on the right-hand side of (5.15) show the change in the long-wave elevation due to the presence of the short wave.

Allowing for differences between $A_2$ and $A_2'$, $\sigma$ and $\sigma_\mathrm{I}$, and $g_1$ in (3.3*c*) and (4.9), we determine the relation between $a_2'$ and $a_2$:

$$a_2' = a_2(1+\tfrac{1}{2}\epsilon_1 a_1 k-(\Omega/\sigma)\,\epsilon_1 a_1 k-\epsilon_1^2). \tag{5.21}$$

Since the difference between them are of third order, we may substitute $a_2$ for $a_2'$ in (5.16) and the changes due to this substitution come only from the leading-order term. As expected, the two short-wave solutions are identical for the case $k > 2K$:

$$\begin{aligned}\zeta = {} & a_2[1-\tfrac{1}{4}a_1^2k^2-\tfrac{3}{8}a_2^2k^2+\epsilon_1 a_1 k(\tfrac{1}{2}-\Omega/\sigma)+\epsilon_1\cos\Theta+(\tfrac{1}{4}a_1^2k^2+\epsilon_1^2)\cos 2\Theta]\cos\theta \\ & +a_2^2 k[\tfrac{1}{2}+\epsilon_1(\tfrac{3}{2}+T_2+\tfrac{1}{4}\epsilon_3)\cos\Theta]\cos 2\theta-a_2 a_1 k(\sin\Theta+\epsilon_1\sin 2\Theta)\sin\theta \\ & +a_1 a_2^2 k^2[-1+\epsilon_3(\tfrac{1}{2}+T_2)]\sin\Theta\sin 2\theta,\end{aligned} \tag{5.22}$$

where $T_2$ is given in (3.11 *b*).

## 6. Conclusion and implications for numerical simulation of wave–wave interactions

The comparison between the solutions derived by two different approaches is not only helpful to check computation, but more importantly, displays the differences in the convergence between the two approaches and the reasons for these differences:

(i) For $\epsilon_3 < \frac{1}{2}$ and $\epsilon_1 \ll \epsilon_3$, both solutions converge rapidly and are identical at least up to $O(\epsilon_{1,2}^3)$. Furthermore, if the computation in both approaches is carried out to an order high enough, their solutions will be identical and converge even if $\epsilon_1 \gg \epsilon_3$, provided that the solution for the individual waves converges.

(ii) When $\epsilon_1 \gg \epsilon_3$, the modulated wave-mode solution converges rapidly, at a rate depending on $O(\epsilon_{1,2})$. On the other hand, the conventional wave-mode solution converges very slowly, at a rate of $O(\epsilon_1^{(m-1)}/(\epsilon_3^{(m-1)}(m-1)!))$, where $m$ is the perturbation order. Hence, the convergence in the conventional approach can be reached only when $m \gg \epsilon_1/\epsilon_3$. In other words, it may diverge if truncated at finite order. This convergence difficulty results from the modelling of a modulated short-wave phase by a linear phase function and all possibly divergent terms originate from the phase difference.

(iii) When $\epsilon_3$ is relatively large, that is $\gg \epsilon_1$, the convergence in the conventional perturbation is reached quickly, at a rate depending on $O(\epsilon_{1,2})$. On the other hand, for $\epsilon_3 \sim O(1)$, part of the solution describing slowly varying wave interaction cannot be accurately predicted by using the modulated wave-mode approach.

The above conclusions have immediate implications for direct time-domain numerical simulations (for example Dommermuth & Yue 1987; West, Brueckner & Janda 1987) of nonlinear wave–wave interactions for general wavefields. Such methods are typically based on the Zakharov equation (Zakharov 1968; Crawford *et al.* 1981) and mode-coupling ideas (for example Phillips 1960; Benney 1962; West, Watson & Thomson 1974; Cohen, Watson & West 1976), but are generalized to include interactions among a large number of conventional wave-modes and to relatively high order in wave steepness. The key step in these calculations is the determination of the vertical velocity at the free surface given its position and the surface potential.

Following Brueckner & West (1988), we examine the convergence of the vertical velocity at the surface for a given short–long wave pair with potential

$$(\Phi+\phi)^{(S)} = (\Omega/K)\,a_1 \sin\Theta + (\sigma/k)\,a_2 \sin\theta, \tag{6.1}$$

on $z = a_1\cos\Theta + a_2\cos\theta$. As in Watson & West (1975) and West (1981), the leading- and second-order vertical velocities can be calculated:

$$W^{(S)(1)} = a_1\,\Omega\sin\Theta + a_2\,\sigma\sin\theta, \tag{6.2}$$

$$W^{(S)(2)} = -\tfrac{1}{2}\epsilon_1\,a_1\,\Omega\sin 2\Theta - \tfrac{1}{2}\epsilon_2\,a_2\,\sigma\sin 2\theta - \epsilon_1(\Omega/\sigma)\,a_2\,\sigma\sin(\theta-\Theta) - \epsilon_1\,a_2\,\sigma\sin\Theta\cos\theta - (\Omega/\sigma)\,a_1\,ka_2\,\sigma\sin\Theta\cos\theta. \tag{6.3}$$

It should be noted that the free-surface boundary conditions are not directly involved in the computation of $W^{(S)}$ and hence the results here are different from those based on (3.14*a*) and (3.15*a*). Regardless of the relation between $\epsilon_1$ and $\epsilon_3$, all terms in $W^{(S)(2)}$ are second order except for the last term. This term has a magnitude which can be comparable to or greater than that of the corresponding leading-order short-wave vertical velocity, $a_2\sigma$, when $O(\epsilon_1^2)$ is comparable to or greater than $O(\epsilon_3)$. Fortunately, this apparent divergence does not present itself at higher orders. For example, at third-order, the corresponding dominant terms in $W^{(S)(3)}$ are of $O(\epsilon_1)$ or $O(\epsilon_2)$ which are smaller than those at second order. A similar result was also reached by using a surface-wave Hamiltonian (Milder 1990). The presence of this large second-order term is likely a root cause of poor numerical performance of existing simulations involving short–long waves using conventional wave-mode functions. It is important to note finally that this large term in the surface vertical velocity when $\epsilon_1^2 \sim \epsilon_3$ does not appear in the modulated wave-mode approach.

As a result of this study, a new time-domain *hybrid* mode-coupling computation scheme is now under development. In this scheme, conventional wave-mode functions are used to model interactions among waves of comparable wavelengths, while modulated wave-mode functions are used to describe short- and long-wave interactions. The computational advantage of this hybrid wave-mode approach for the simulation of a realistic wave field is being explored and will be reported in the future.

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## Appendix A. Derivation of governing equations and scale factor

### A.1. *Scale factor*

By the definition of the scale factors,

$$H_1(s,n) = \mathrm{d}s/\mathrm{d}s', \quad H_2(s,n) = \mathrm{d}n/\mathrm{d}n', \tag{A 1}$$

where $\mathrm{d}s$, $\mathrm{d}n$, $\mathrm{d}s'$ and $\mathrm{d}n'$ are respectively the increments in the $(s,n)$-coordinates and their projections in the $(x',z)$-plane (see figure 1). Using the definition of the conformal mapping and the relationship between the flow rate and the stream function,

$$\mathrm{d}n = -\mathrm{d}\Psi'/|C| = U(x',z)\,\mathrm{d}n'/C,$$

and similarly, the relationship between the potential function and the velocity,

$$\mathrm{d}s = -\mathrm{d}\Phi'/|C| = U(x',z)\,\mathrm{d}s'/C,$$

we obtain

$$H = H_1 = H_2 = U(x',z)/C. \tag{A 2}$$

Using the relation of the conformal mapping we finally obtain (2.14).

### A.2. *Governing equations in the* $(s,n)$*-plane*

The governing equations in the $(s,n)$-plane may be obtained either directly or by transforming their counterparts in the $(x',z)$-plane. Both approaches yield identical results. For simplicity, we show the derivation of the governing equations through transformation except for the free-surface kinematic boundary condition.

For incompressible flows,

$$\frac{\partial}{\partial s'}\left[\left(\frac{\partial\phi}{\partial s'}+U\right)\mathrm{d}n'\right]+\frac{\partial}{\partial n'}\left[\frac{\partial\phi}{\partial n'}\mathrm{d}s'\right] = 0, \quad -\infty < z < \zeta+\eta, \tag{A 3}$$

where $\partial\phi/\partial s'+U$ and $\partial\phi/\partial n'$ are velocities in the direction of streamlines and equipotentials, respectively. Using (A 1) and (A 2), (A 3) reduces to (2.9). In the moving coordinates $(x',z)$, the long wave is steady. Also noting that

$$\eta(x)+\zeta(x,t) = \eta(x_0)+\xi'(x_0)\cos\alpha, \tag{A 4}$$

(2.2) can be transformed to obtain (2.10) and (2.13). The transformation for (2.12) is trivial.

To derive the kinematic boundary condition, we map the velocity in the $(x',z)$- to the $(s,n)$-plane. Since the conformal mapping is time independent, the relationship for the velocity is

$$u(s,n) = H((\partial\phi/\partial s')+U) = H^2\,\partial\phi/\partial s+HU, \tag{A 5}$$

$$w(s,n) = H\,\partial\phi/\partial n' = H^2\,\partial\phi/\partial n. \tag{A 6}$$

Applying the material derivative to the free surface in the $(s,n)$-plane, we have

$$w(s,n) = (\partial\xi/\partial t)+(\partial\xi/\partial s)\,u(s,n) \qquad \text{at} \quad n = \xi(s,t). \tag{A 7}$$

With (A 5) and (A 6), (A 7) reduces to (2.11).

## Appendix B. Derivation of $\mathscr{L}$ in (4.4)

Using complex form, setting $A_2' = 1$ for simplicity and noticing that

$$\nabla^2 = \left(\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial n^2}\right),$$

then
$$\nabla^2(-\mathrm{i}\,\mathrm{e}^{k'n+\mathrm{i}\theta'}) = \left[\frac{\partial k'}{\partial s}+2k'\frac{\partial k'}{\partial s}n-\mathrm{i}\frac{\partial^2 k'}{\partial s^2}-\mathrm{i}\left(\frac{\partial k'}{\partial s}\right)^2 n^2\right]\mathrm{e}^{k'n+\mathrm{i}\theta'}. \tag{B 1}$$

According to (5.6), the first three terms in the brackets of (B 1) are of $O(\epsilon_1)k'^2$ and the last term of $O(\epsilon_1^2)k'^2$. This indicates that the Laplacian of $\mathrm{e}^{k'n+\mathrm{i}\theta'}$ is of $O(\epsilon_1)k'^2\,\mathrm{e}^{k'+\mathrm{i}\theta'}$.

Consider the expansion of $\mathscr{L}$ with respect to $\epsilon_1$, $\mathscr{L} = \sum_{j=1}^{\infty}\mathscr{L}_j$, where the subscript $j$ denotes the order in $\epsilon_1$. $\mathscr{L}_j$ can be obtained sequentially for $j = 1, 2, \ldots$ from

$$\nabla^2\left(-\mathrm{i}\,\mathrm{e}^{k'n+\mathrm{i}\theta'}+\sum_{l=1}^{j-1}\mathscr{L}_l+\mathscr{L}_j\right) = O(\epsilon_1)^{j+1}k'^2\,\mathrm{e}^{k'n+\mathrm{i}\theta'}. \tag{B 2}$$

Thus
$$\mathscr{L}_1 = \sum_{m=2}^{\infty}\frac{(-\mathrm{i})^m}{m!}\frac{\partial^{m-1}k'}{\partial s^{m-1}}n^m\,\mathrm{e}^{k'n+\mathrm{i}\theta'},\quad \mathscr{L}_2 = \sum_{m=4}^{\infty}D_m n^m\,\mathrm{e}^{k'n+\mathrm{i}\theta'},\ldots, \tag{B 3}$$

where
$$D_4 = \tfrac{1}{8}\mathrm{i}(\partial k'/\partial s)^2, \tag{B 4}$$

$$D_m = -\frac{\mathrm{i}}{m}\frac{\partial D_{m-1}}{\partial s}+(-\mathrm{i})^{m-1}\frac{m-1}{m!}\frac{\partial k'}{\partial s}\frac{\partial^{m-3}k'}{\partial s^{m-3}}\quad\text{for}\quad m\geqslant 5. \tag{B 5}$$

If $\mathscr{L}$ is truncated at $j = 2$, the omitted terms are of $O(\epsilon_1^3)\,\mathrm{e}^{k'n+\mathrm{i}\theta'}$. For relatively small $\epsilon_3$, the solution may be reduced to

$$\mathscr{L} = A_2'\left\{-\tfrac{1}{2}n^2\frac{\partial k'}{\partial s}\cos\theta'\,\mathrm{e}^{k'n}-\frac{1}{3!}n^3\frac{\partial^2 k'}{\partial s^2}\sin\theta'\,\mathrm{e}^{k'n}\right.$$
$$\left.+\frac{1}{4!}n^4\frac{\partial^3 k'}{\partial s^3}\cos\theta'\,\mathrm{e}^{k'n}-\tfrac{1}{8}n^4\left(\frac{\partial k'}{\partial s}\right)^2\cos\theta'\,\mathrm{e}^{k'n}\right\}, \tag{B 6}$$

which is consistent with Zhang (1991).

## Appendix C. Solution of the short-wave second harmonic

The boundary conditions at $n = 0$ for the second harmonic of the short wave can be derived from (4.1) and (4.2):

$$\phi_t^{(2)}+H_0^2C\phi_s^{(2)}+g\zeta^{(2)} = P^{(2)}, \tag{C 1}$$
$$\zeta_t^{(2)}+H_0^2C\zeta_s^{(2)}-H_0^2\phi_n^{(2)} = Q^{(2)}, \tag{C 2}$$

where $P^{(2)}$ and $Q^{(2)}$ denote the forcing terms resulting from the first harmonic:

$$P^{(2)} = -\frac{\partial}{\partial n}(\phi_t^{(1)}+H_0^2C\phi_s^{(1)})\,\zeta^{(1)}-\tfrac{1}{2}H_0^2(\phi_s^{(1)^2}+\phi_n^{(1)^2})+O(\epsilon_{1,2}^3)A_2'\sigma_1, \tag{C 3}$$

$$Q^{(2)} = -H_0^2\frac{\partial}{\partial s}(\phi_s^{(1)}\zeta^{(1)})-\frac{\partial H_0^2}{\partial n}C\zeta^{(1)}\zeta_s^{(1)}+\frac{\partial H_0^2}{\partial n}\phi_n^{(1)}\zeta^{(1)}\sigma_{\mathrm{I}}+O(\epsilon_{1,2}^3)a_2'\sigma_{\mathrm{I}}. \tag{C 4}$$

Using (4.4)–(4.9), (5.2) and (5.5), they reduce to

$$P^{(2)} = \tfrac{1}{2}a_2'^2k'g_1\cos 2\theta'-(\Omega/\sigma_{\mathrm{I}})\,\epsilon_1 a_2'^2k'g_1\cos(2\theta'-Ks), \tag{C 5}$$
$$Q^{(2)} = a_2'^2k'\sigma_{\mathrm{I}}\sin 2\theta'+(\Omega/\sigma_{\mathrm{I}})\,\epsilon_1 a_2'^2k'\sigma_{\mathrm{I}}\cos Ks\sin 2\theta'-\epsilon_1 a_2'^2K\sigma_{\mathrm{I}}\sin(2\theta'-Ks). \tag{C 6}$$

The derivation of the second-harmonic solution involves an additional perturbation in terms of $\epsilon_3^{\frac{1}{2}}$ and is very lengthy. The solution up to $O(\epsilon_{1,2}^3)$ is given by

$$\phi^{(2)} = \sum_{m=1}^{\infty} \frac{m+2}{2}\left(\frac{\Omega}{\sigma_I}\right)^m \epsilon_1 a_2' k' A_2' \mathrm{e}^{(2k'-K)n} \sin(2\theta' - Ks) + O(\epsilon_{1,2}^3) A_2', \tag{C 7}$$

$$\xi^{(2)} = \tfrac{1}{2} a_2'^2 k' \cos 2\theta' + \sum_{m=1}^{\infty} \frac{m+3}{2}\left(\frac{\Omega}{\sigma_I}\right)^m \epsilon_1 a_2'^2 k' \cos(2\theta' - Ks) + O(\epsilon_{1,2}^3) a_2'. \tag{C 8}$$

Noting that $\Omega/\sigma_I < 1$ and making use of the geometry series, we simplify (C 7) and (C 8) to (4.10) and (4.11), respectively. It is clear that the potential (4.10) satisfies the Laplace equation (up to the third order) and the bottom boundary condition. Using (5.5), $CK = -\Omega$ and $K/k' = (\Omega/\sigma_I)^2$, we may also show that (4.10) and (4.11) satisfy (C 1) and (C 2).

When $\epsilon_3$ is small ($\sim O(\epsilon_1^{\frac{1}{2}})$), (C 7) and (C 8) can be truncated and reduce to

$$\phi^{(2)} = \tfrac{3}{2}(\Omega/\sigma_I)\, \epsilon_1 a_2' k' A_2' \mathrm{e}^{2k'n} \sin(2\theta' - Ks), \tag{C 9}$$

$$\xi^{(2)} = \tfrac{1}{2} a_2'^2 k' \cos 2\theta + 2(\Omega/\sigma_I)\, a_2'^2 k' \cos(2\theta' - Ks), \tag{C 10}$$

## Appendix D. Average short-wave frequency through the modulated wave-mode approach

It takes time $t_0$ for the short wave to sweep backward one long wavelength in the moving coordinates:

$$t_0 = \int_0^{-2\pi/K} \frac{\mathrm{d}s'}{(U + \sigma_I/k')}. \tag{D 1}$$

Using (A 1), (A 2), (5.5) and (5.2), we have

$$t_0 = \int_0^{2\pi/K} \frac{\mathrm{d}s}{H_0^2[(\Omega/K) - (\sigma_I/k_0')]} = \frac{2\pi(1+\epsilon_1^2)}{K[(\Omega/K) - (\sigma_I/k_0')]}. \tag{D 2}$$

The advance of the short wave in the fixed coordinates is given by

$$x_0 = (\Omega/K)\, t_0 - 2\pi/K. \tag{D 3}$$

Hence, the average phase velocity of the short wave is

$$x_0/t_0 = (\sigma_I/k) + \epsilon_1^2(\Omega/K) + O(\epsilon_{1,2}^3)\, \sigma_I/k, \tag{D 4}$$

and its average frequency is $\sigma = \sigma_I + a_1^2 kK\Omega$ which is identical to (3.8).

## Appendix E. Conversion of the short-wave potential

The direct conversion of $\mathscr{L}$ in (4.4) is cumbersome. To avoid this, we impose a Dirichlet boundary-value problem formulated by:

$$\nabla^2\phi = 0, \quad -\infty \leqslant n \leqslant 0, \tag{E 1a}$$

$$\phi = \phi^{(1)} = A_2' \sin\theta' \quad \text{at} \quad n = 0, \tag{E 1b}$$

$$\phi \to 0 \quad \text{as} \quad n \to -\infty, \tag{E 1c}$$

$$\partial\theta'/\partial s|_{s=0} = \partial\theta'/\partial s|_{s=2\pi/K}. \tag{E 1d}$$

The lateral boundary condition (E 1$d$) requires that the short wavenumber be modulated periodically along the long wave. The Dirichlet problem is formulated so that $\phi^{(1)}$ in (4.4) satisfies it. Since the solution to the Dirichlet problem must be unique, we seek an alternative form of the solution which is easier to convert. Let

$$\phi = A_2' \sum_{m=0}^{\infty} B_m \, e^{|k \pm mK|n} \sin(\theta_0' \pm mKs), \tag{E 2}$$

where $\theta_0' = k_0'(1+\epsilon_1^2)\,s+\beta = ks+\beta$. Equation (E 2) satisfies the Laplace equation, bottom and lateral boundary conditions. The surface boundary condition, (E 1$b$), is used to determine the coefficients $B_m$:

$$B_0 = 1 - a_1^2 k^2, \quad B_{\pm 1} = \pm a_1 k, \quad B_{\pm 2} = \tfrac{1}{2} a_1^2 k^2 \pm \epsilon_1 a_1 k, \quad B_{\pm m} = O(a_1^3 k^3).$$

To ensure agreement between the original solution $\phi^{(1)}$ and (E 2) truncated at $m = 2$, the truncation error must be small, that is $\epsilon_1 \ll \epsilon_3$. As for the conversion of the wave elevation, this truncation error results from the phase difference.

Substituting (2.6$a$) and (2.6$b$) into (E 2), we may transform $(s, n)$ in the potential to $(x, z)$ and derive (5.17). Again $\epsilon_1 \ll \epsilon_3$ is required for the validity of the truncation.

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